

QCD Corrections for the $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ System

A. Datta, E.A. Paschos, J.-M. Schwarz

Institut fuer Physik, Universitaet Dortmund
D-44221 Dortmund, Germany

and

M.N. Sinha Roy

Dept. of Physics, Presidency College
Calcutta 700 073, India

Abstract

The QCD corrections for the box diagrams are revisited for the case of a heavy top quark with $m_t = 174 GeV$. We resolve first a long-standing discrepancy between two methods of calculation by showing that they give the same results when the threshold factors are treated correctly. Using this observation we refine our earlier results and derive formulae valid for the K - and B -meson systems. Our formulae are given in terms of integrals to be evaluated numerically, as well as approximate analytical formulae. These calculations include the evolution above m_W which has been neglected by other authors.

1 Introduction

Perturbative quantum chromodynamic (QCD) corrections to electroweak processes, dominated by short distance physics, were calculated in the leading logarithmic approximation (LLA) long time ago [1]. The pioneering calculations studied the enhancement of the $\Delta I = 1/2$ -amplitude in $\Delta S = 1$ processes and the origin of direct CP violation.

Subsequently, QCD corrections in the LLA were computed for $\Delta S = 2$ processes [2]-[4] assuming that all the quarks in the loops are much lighter than the W bosons ($m_q \ll m_W$ for all q). In course of time, the gradual strengthening of the lower limit on the mass m_t of the t quark lead to the realization that m_t may indeed be comparable to m_W or even larger. This motivated several papers which generalize the calculations for a heavy top quark [5]-[9]. Some of these articles [6]-[9] simplified the calculation by neglecting the evolution of the Wilson coefficients in the interval $m_W \rightarrow m_t$. This approximation was not made in [5]. However, since m_t was unknown, the numerical significance of this approximation could not be assessed properly.

In the meanwhile, the production of the top quark has been reported by two Fermilab groups. The CDF collaboration [10] finds a top quark with a mass $m_t = 174 \pm 10 \pm {}^{13}_{12} GeV$ and the D0 collaboration [11] with a mass $m_t = 180 \pm 12 GeV$. These values are in good agreement with the precision measurements at LEP-I where it was established [12, 13] (albeit indirectly) that $m_t = 169 \pm {}^{16}_{18} \pm {}^{17}_{20} GeV$. All this forces us to accept that the mass difference between the top quark and the W-boson is indeed significant. It is therefore worthwhile to revisit the calculations of [5] and compare them with other results [6]-[8] in order to estimate the numerical significance of the evolution beyond m_W . We find that this evolution is numerically significant for $m_t \approx 174 GeV$ although it does not change the result drastically.

The first issue concerns the method of calculation to be adopted in this article. There are two methods : the first one is diagrammatic where one loop QCD corrections are computed on top of the weak box diagram. The lowest order corrections thus obtained, are summed up using the renormalisation group (RG). The resulting Wilson coefficients are dependent on the momentum of the electroweak loop (p^2) which are then integrated over p^2 [2, 4, 5]. The second method first introduced in [3] computes the QCD corrections in

a series of effective field theories obtained by successively integrating out the heavy degrees of freedom [3, 6, 7, 8]. The two methods have a long-standing and unsettled issue which was raised in ref. [3]. To be specific, even in case of $m_t \ll m_W$, the results of [3] (which heavily uses the rather involved techniques of operator mixing) and ref. [4] did not agree. Indeed it was noted in ref. [3] that although two of the three QCD correction factors (η_1, η_2) computed by them agree with ref. [4] after appropriate simplifications, the third (η_3) did not (see footnote 14 of ref. [3]). To the best of our knowledge this discrepancy has not been clarified in the subsequent literature. Since the disagreement was precisely in a term where operator mixings played an important rôle in ref. [3], a critical reader may also question the procedure of ref. [4], which did not explicitly use operator mixing. This critique would also apply to ref. [5] where the techniques of Novikov, Shifman, Vainshtein, Zakharov [2] and Visotskiĭ [4] (NSVZV) was generalized for $m_t \gg m_W$.

In this paper we explicitly demonstrate that the apparent disagreement between ref. [3] and [4] is a consequence of certain simplifying assumptions in ref. [4]. More specifically in ref. [4] certain Wilson coefficients were evolved over large momentum intervals which involve several thresholds. Some remarks were made regarding the ambiguities in determining the number of active quark flavours (n_f) in these intervals, but for the purpose of actual computation, n_f was kept fixed. We find that by changing n_f at each new quark threshold and taking the matching conditions at each threshold [4] into account a more rigorous formula can be derived from the procedure of ref. [4]. This leads to an agreement — not only numerically but also algebraically — with ref. [3].

Although the case $m_t \ll m_W$ is now of academic interest only, we consider the above demonstration important because it illustrates that both methods of calculation (GW and NSVZV and its generalization [5]) are equivalent. The apparent disagreement comes from the approximate treatment of the threshold factors as discussed above. In the present article, we remove these approximations and sharpen the formulae of ref. [5]. In section 3 we present explicit and improved formulae which are in one-to-one correspondence to our previous work [5]. The formulae by themselves answer several questions raised about our earlier work (see in particular [8]).

The main new result of ref. [5] was to demonstrate that a particular Wilson coefficient computed in [2, 4] does not contribute if $m_t \gg m_W$. The remaining Wilson coefficients were calculated. The p^2 dependent Wilson

coefficients as discussed above were integrated numerically over the entire permissible range of p^2 . In this paper we incorporate the threshold factors discussed earlier, into these integrations and obtain very accurate results.

However, from the final results thus obtained the numerical significance of the evolution of $\alpha_s(p^2)$ over a specific interval ($m_W \rightarrow m_t$) cannot be isolated automatically. This can be done by separating the regions of the numerical integration and setting $\alpha_s(p^2) = \alpha_s(m_W^2)$ for $p^2 > m_W^2$. Alternatively, in order to separate the evolution of this specific interval, we introduce a reasonable approximation, which allows us to perform the p^2 -integration analytically. This way we isolate the interval ($m_W \rightarrow m_t$), where we can choose $\alpha_s(p^2)$ as either running or as a constant with $\alpha_s(p^2) = \alpha_s(m_W^2)$ for $p^2 > m_W^2$. We thus obtain for the first time a set of analytical formulae which explicitly exhibits the effects of the evolution above m_W and readily reduces to the results of [6, 7, 8] once these evolutions are neglected. However, as we summarize below in various tables, the evolution above m_W , though not completely negligible, does not change the results drastically. This is partly due to the fact that various corrections cancel each other. Finally, the formalism of this article is very transparent and general so that it can be easily extended to four generations, where higher scales become relevant. In such cases the evolution above m_W can in principle be crucially important. It has been shown that models with naturally heavy Majorana [14] or Dirac [15] neutrinos belonging to the fourth generation can be constructed. Such models are, therefore, neither unnatural nor in conflict with the neutrino counting at LEP [12, 13].

In ref. [8] next to leading order calculations for one of the QCD factors, η_2 in the notation of [3], have been performed. These attempts are important since they have the potential of settling important theoretical issues, like the independence of the final result from the scale at which the initial conditions for the Wilson coefficients have been imposed or from the definitions of the heavy top masses, etc. Such refinements however can attain their full potential only if the corrections in the LLA are known as accurately as possible. In this paper we find that in some cases the evolution above m_W is numerically of the same order as the nonleading corrections and requires, therefore, a closer scrutiny (see below). This explains, in part, the fact that the authors in [8] obtained for η_2 the same result. For the case $m_t > m_W$ there is for η_3 only the calculation in [5] and the present one.

The paper is organized as follows. In section 2 we briefly review the

well known basic formulae for $\Delta F = 2$ processes relevant for $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ mixings in order to set up the notation and make the article self-contained. We then review the computation of the QCD corrections in the hypothetical case $m_t \ll m_W$ following the NSVZV method and demonstrate the equivalence with GW. This is reached by treating the threshold factors and the matching conditions carefully. Using the experience gained in section 2, we refine the results of ref. [5] and present them as specific integrals. This is done explicitly for each integral of the effective weak Hamiltonian and summarized in section 3. In addition, section 3 includes analytic formulae obtained after integrating the terms, in a certain approximation, in order to separate the evolution above m_W and compare it with other works [6] – [8]. The formulae look long and complicated, but are obtained by a straight-forward treatment of thresholds. In section 4 we evaluate the results numerically arriving at final values for the QCD factors and compare them with those in other articles.

2 QCD Corrections in the NSVZV approach

In this section we consider the hypothetical case $m_W \gg m_q$ (for all q) in order to connect with the early work on this subject and in particular to demonstrate the equivalence between the works of Gilman and Wise [3] and Visotskiĭ [4].

In this limit ($m_W \gg m_q$) the effective Hamiltonian without QCD corrections is obtained from the box diagram (Fig. 1) with two W boson exchanges and attains the form [16]:

$$\begin{aligned}
H_{eff} &= \frac{G_F^2 m_W^2}{16\pi^2} (\bar{d}\gamma_\mu(1 - \gamma_5)s)^2 \sum_{i,j=c,t} \lambda_i \lambda_j [I(m_i^2, m_j^2) - I(m_i^2, 0) - I(0, m_j^2) + I(0, 0)] \\
&\equiv \frac{G_F^2 m_W^2}{16\pi^2} (\bar{d}\gamma_\mu(1 - \gamma_5)s)^2 \sum_{i,j=c,t} \lambda_i \lambda_j H_{eff}(i, j)
\end{aligned} \tag{1}$$

where the GIM subtraction is explicitly introduced with $m_u \approx 0$. We follow the standard notation $\lambda_i = V_{id}^* V_{is}$, where the V 's are the elements of the Cabibbo-Kobayashi-Maskawa (CKM) matrix. The dimensionless integrals $I(m_i^2, m_j^2)$ are over Euclidean loop momentum,

$$I(m_i^2, m_j^2) = \int \frac{dp^2 p^4 m_W^2}{(p^2 + m_i^2)(p^2 + m_j^2)(p^2 + m_W^2)^2} \quad (2)$$

These integrals are convergent so that the dominant contributions come from $p^2 \approx m_i^2, m_j^2$ as well as $p^2 \approx m_W^2$. Due to the GIM subtraction in Eq. (1) the contribution from the region $p^2 \approx m_W^2$ cancels out. Hence it is justified to replace the W propagators by m_W^{-2} . Now H_{eff} is implicitly the time-ordered product of two $\Delta S = 1$, local four fermion operators. This of course can be seen most transparently in coordinate space (see Eqs. 5a–5c of ref. [3]). These operators can be written as linear combinations of the well known colour symmetric and antisymmetric operators O^\pm . The QCD corrections for these multiplicatively renormalisable operators are well known [1]. Finally the QCD corrected $\Delta S = 1$ operators (neglecting mixings with penguins) are joined again by Wick's rule. In momentum space this corresponds to the replacement of the W boson propagators in Eq. (2) by m_W^{-4} and multiplying each integral of Eq. (1) by a factor [2, 4]

$$Q_1(m_W^2, p^2, n_1) = \frac{1}{2} \left(\frac{\alpha_{n_1}(m_W^2)}{\alpha_{n_1}(p^2)} \right)^{\frac{2a_-}{b(n_1)}} - \left(\frac{\alpha_{n_1}(m_W^2)}{\alpha_{n_1}(p^2)} \right)^{\frac{a_+ + a_-}{b(n_1)}} + \frac{3}{2} \left(\frac{\alpha_{n_1}(m_W^2)}{\alpha_{n_1}(p^2)} \right)^{\frac{2a_+}{b(n_1)}} \quad (3)$$

where $a_+ = 2$, $a_- = -4$ are the anomalous dimensions of O^\pm in units of $\alpha_{n_1}/(2\pi)$, α_{n_1} is the strong coupling constant and $b(n_1) = 11 - \frac{2}{3}n_1$ depends on the number of massless quarks in the interval $m_W \rightarrow p^2$. We emphasize that since p^2 is variable n_1 changes as p^2 crosses various quark thresholds.

The two momentum scales in Eq. (3) are worth noting. It was emphasized in ref. [2] that for Green functions involving loops, p^2 provides an important scale and the emergence of momentum (p^2) dependent Wilson coefficients was explicitly demonstrated (see in particular the third article of ref. [2]). Moreover, as pointed out in ref. [2, 4], and directly shown in ref. [5], the $O(\alpha)$ terms in the expansion of Eq. (3) can be directly identified with the diagrams of figure 1 (see Eqs. 20–21b of ref. [5]).

Once p^2 becomes larger than the masses and momenta on the external legs, the effective four fermion operator $(\bar{d}\gamma_\mu(1 - \gamma_5)s)^2$ is generated. This property is already implicit in Eq. (1), which is derived in the approximation of neglecting the external momenta and it also holds in the realistic case $m_t > m_W$. The evolution of the four fermion operator $(\bar{d}\gamma_\mu(1 - \gamma_5)s)^2$, which is colour symmetric and has the same anomalous dimension as O^+ , from p^2

down to μ^2 (a scale where perturbative QCD presumably breaks down) yields a factor

$$Q_2(p^2, \mu^2, n_2) = \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{\frac{2}{b(n_2)}} \quad (4)$$

which multiplies the integrands of Eq. (1). The $O(\alpha)$ term in the expansion of Eq. (4) can be explicitly checked from fig. 2 as was done in ref. [5] (see Eq. (12) of [5] and the discussion following it). It is very important to note here that the number of active flavours (n_2) in Eq. (4) (which is not fixed as yet) can be different from that in Eq. (3) since the momentum intervals in the two cases are different.

Finally each of the heavy quarks in the internal lines are off mass-shell during the p^2 integration and can be dressed with gluons as in figure 3. This corresponds to replacing each heavy quark mass in Eq. (1) by [2, 4]

$$\begin{aligned} m_q^2(p^2) &= m_q^2(m_q^2) \left(1 + b(n_3) \frac{\alpha_{n_3}(m_q^2)}{4\pi} \ln \frac{p^2}{m_q^2} \right)^{-8/b(n_3)} \\ &= m_q^2(m_q^2) \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_q^2)} \right)^{8/b(n_3)} \end{aligned} \quad (5)$$

which is the well known relation between running quark masses at scales p^2 and m_q^2 . The right-hand side of Eq. (5) was not shown explicitly in ref. [2, 4, 5], but was evidently defined correctly. This caused some criticism in the subsequent literature [8], which is not justified.

Following our above observations we have kept the number of flavours (n_3 in Eq. (5)) arbitrary and in principle different from those in Eqs. (3,4). It is rather obvious that this correction is also independent of the condition $m_W^2 \gg m_t^2$ and continues to hold in the realistic case $m_t > m_W$.

The removal of the W boson propagators and GIM subtraction leads to a generic contribution to the effective Hamiltonian defined in Eq. (1) (in the absence of QCD corrections)

$$H_{eff}(i, j) \approx \frac{1}{m_W^2} \int \frac{m_i^2 m_j^2 dp^2}{(p^2 + m_i^2)(p^2 + m_j^2)}. \quad (6)$$

When two quarks on the internal lines of the box diagram are identical ($i = j = c$ or t , henceforth referred to as the cc or tt graph) the integral is dominated by contributions from $p^2 \approx m_i^2$ [2, 4]. On the other hand, for the

diagram with one internal c quark and one t quark (the ct graph), the entire region $m_c^2 \leq p^2 \leq m_t^2$ contributes as can be seen from the following form [2, 4]

$$\begin{aligned} H_{eff}(c, t) &\approx \frac{m_c^2 m_t^2}{m_W^2 (m_t^2 - m_c^2)} \int_{m_c^2}^{m_t^2} \frac{dp^2}{p^2} \\ &\approx x_c \ln \frac{m_t^2}{m_c^2} \end{aligned} \quad (7)$$

where $x_q = m_q^2/m_W^2$.

Due to QCD corrections the above formula should be modified to

$$H_{eff}(i, j) = \frac{1}{m_W^2} \int \frac{dp^2 m_i^2(p^2) m_j^2(p^2) Q_1(m_W^2, p^2, n_1) Q_2(p^2, \mu^2, n_2)}{(p^2 + m_i^2(p^2))(p^2 + m_j^2(p^2))} \quad (8)$$

where the corrections are defined in Eqs. (3)–(5). A very accurate analytical result can be obtained by the following approximation. It is assumed that p^2 dependent QCD corrections cannot drastically alter the important regions of momentum integration as noted after Eq. (6). For the $i = j$ case, this corresponds to evaluating the corrections (Eqs. 3–5) at $p^2 = m_i^2(m_i^2)$ and one obtains from Eq. (8)

$$H_{eff}(i, i) = x_i Q_1(m_W^2, m_i^2, n_1) Q_2(m_i^2, \mu^2, n_2) \quad (9)$$

For notational convenience we shall denote henceforth $m_i(m_i^2) = m_i$.

In ref. [4], n_2 in Eq. (4) was assumed to be 3 for numerical calculations. While this is justified for the cc graph the same cannot be said about the tt graph. Similarly n_1 was chosen to be 4 which is not rigorously justified either for the cc or for the tt graph. For the cc graph, for example, one can replace a typical term in Q_1 of Eq. (9) by

$$\left(\frac{\alpha_{n_1}(m_W^2)}{\alpha_{n_1}(m_c^2)} \right)^{a/b(n_1)} \longrightarrow \left(\frac{\alpha_6(m_W^2)}{\alpha_6(m_t^2)} \right)^{a/b(6)} \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_b^2)} \right)^{a/b(5)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{a/b(4)} \quad (10)$$

where the changes in the number of flavours at different thresholds have been explicitly included. It is further assumed that the matching conditions as defined in [3] are also obeyed. With this modification Eq. (9) reproduces the results of ref. [3], which was already noted in ref. [3]. A similar agreement

for the tt graph can be obtained analogously by modifying the Q_2 term in Eq. (9).

The situation is somewhat more complicated for the ct graph. Using Eq. (7), we can write the integral of Eq. (8) in the following approximate form

$$H_{eff}^{QCD}(c, t) = \frac{1}{m_W^2} \int_{m_c^2}^{m_t^2} \frac{dp^2}{p^2} m_c^2(p^2) Q_1(m_W^2, p^2, n_1) Q_2(p^2, \mu^2, n_2) \quad (11)$$

When we substitute in Eqs. (3)–(5) the number of flavours $n_{1,2,3} = 4, 3, 4$, respectively, and factorize

$$\frac{\alpha(p^2)}{\alpha(\mu^2)} = \frac{\alpha(p^2)}{\alpha(m_c^2)} \frac{\alpha(m_c^2)}{\alpha(\mu^2)}$$

and

$$\left(\frac{\alpha(m_W^2)}{\alpha(p^2)}\right) = \left(\frac{\alpha(m_W^2)}{\alpha(m_c^2)}\right) \left(\frac{\alpha(m_c^2)}{\alpha(p^2)}\right)$$

which roughly corresponds to the approximation that above m_c the number of active quark flavour is fixed at 4, the integration in Eq. (11) can be done easily and η_3 of ref. [4] is rederived. The resulting formula unfortunately disagrees with ref. [3]. This is the disagreement already mentioned and will be resolved below.

In order to treat the number of active flavours consistently, we note that a typical term of Eq. (11) is of the following form:

$$I_a = \int_{m_c^2}^{m_t^2} \frac{dp^2}{p^2} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)}\right)^{(8/b(n_3))} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)}\right)^{(2/b(n_2))} \left(\frac{\alpha_{n_1}(p^2)}{\alpha_{n_1}(m_W^2)}\right)^{(a/b(n_1))} \quad (12)$$

Values for n_1 , n_2 , and n_3 can be assigned consistently by splitting the integral into two pieces $m_c^2 \leq p^2 \leq m_b^2$ and $m_b^2 \leq p^2 \leq m_t^2$. Now each momentum interval can be assigned a well defined number of active flavours. This yields

$$\begin{aligned} I_a = & \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)}\right)^{(2/b(3))} \left(\frac{\alpha_6(m_t^2)}{\alpha_6(m_W^2)}\right)^{(a/b(6))} \\ & \left\{ \left(\frac{\alpha_5(m_b^2)}{\alpha_5(m_t^2)}\right)^{a/b(5)} \left(\frac{\alpha_4(m_c^2)}{\alpha_4(m_b^2)}\right)^{a/b(4)} \int_{m_c^2}^{m_b^2} \frac{dp^2}{p^2} \left(\frac{\alpha_4(p^2)}{\alpha_4(m_c^2)}\right)^{(8+a+2)/b(4)} \right. \\ & \left. + \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)}\right)^{10/b(4)} \left(\frac{\alpha_5(m_b^2)}{\alpha_5(m_t^2)}\right)^{a/b(5)} \int_{m_b^2}^{m_t^2} \frac{dp^2}{p^2} \left(\frac{\alpha_5(p^2)}{\alpha_5(m_b^2)}\right)^{(10+a)/b(5)} \right\} \quad (13) \end{aligned}$$

where $a = -4, 2$ or 8 and replacements similar to Eq.(10) have been made.

Using the general formula

$$\begin{aligned} J(m_2^2, m_1^2, x) &= \int_{m_1^2}^{m_2^2} \frac{dp^2}{p^2} \left(1 + \frac{b(n)}{4\pi} \alpha_n(m_1^2) \ln \frac{p^2}{m_1^2} \right)^{-x} \\ &= \frac{4\pi}{b(n)\alpha_n(m_2^2)(1-x)} \left(\frac{\alpha_n(m_2^2)}{\alpha_n(m_1^2)} \right)^x \left(1 - \left(\frac{\alpha_n(m_1^2)}{\alpha_n(m_2^2)} \right)^{x-1} \right) \end{aligned} \quad (14)$$

the integrals in Eq.(13) can be done analytically.

In simplifying the results, the matching conditions (e. g. $\alpha_4(m_b^2) = \alpha_5(m_b^2)$ [3]) should be used in particular for the $\alpha_n(m_2^2)$ in the overall factor in Eq. (14). This leads to, e.g.

$$\alpha_5(m_t^2) = \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_b^2)} \right) \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right) \alpha_4(m_c^2) \quad (15)$$

Using all of the above one finally obtains

$$\begin{aligned} I_a &= \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_6(m_t^2)}{\alpha_6(m_W^2)} \right)^{a/b(6)} \frac{4\pi}{\alpha_4(m_c^2)(b(4) - 10 - a)} \\ &\quad \left[\left(1 - \frac{b(4) - 10 - a}{b(5) - 10 - a} \right) \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{10/b(4)-1} \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_b^2)} \right)^{-a/b(5)} \right. \\ &\quad \left. - \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_b^2)} \right)^{-a/b(5)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{-a/b(4)} \right. \\ &\quad \left. + \left(\frac{b(4) - 10 - a}{b(5) - 10 - a} \right) \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{10/b(4)-1} \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_b^2)} \right)^{10/b(5)-1} \right] \end{aligned} \quad (16)$$

In this notation Eq. (11) can be rewritten as

$$H_{eff}(c, t) = x_c \left[\frac{1}{2} I_8 - I_2 + \frac{3}{2} I_{-4} \right] \quad (17)$$

which is algebraically identical to Eq. (23) of ref. [3]. In summary we reiterate that the effects of the mixing of the operators O_7, O_8 in the coordinate space as studied in ref. [3], can be equivalently treated in this momentum space calculation by handling the p^2 integration properly (notice the mixing of the anomalous dimensions of various operators in the integrands of Eq. (13)). Consequently, one can use the methods of GW or NSVZV and obtain the

same results, provided that threshold effects are handled appropriately. In the next section we include these effects in the approach of NSVZV in order to refine the results of ref. [5] for the case of $m_t \gg m_W$.

3 QCD corrections in LLA for $m_t \gg m_W$

We now present the QCD correction factors for the realistic case $m_t \gg m_W$ which is a refined version of the calculations of ref. [5] along the lines developed in section 2. In this calculation diagrams with unphysical Higgs scalar exchanges are also important. In the following we denote by the subscripts WW, WH and HH the contributions of diagrams with two W boson internal lines, one W and one Higgs boson internal lines and two Higgs boson internal lines respectively. For the cc diagram the approximation $m_W \gg m_c$ is still justified. Thus the results of ref. [3, 4] are still valid.

For the ct graph the terms in Eq. (1) can be regrouped in the following way

$$H_{WW}^a(c, t) = [I(0, 0) - I(0, m_c^2)] \quad (18)$$

$$H_{WW}^b(c, t) = [I(m_c^2, m_t^2) - I(0, m_t^2)] \quad (19)$$

In Eq. (18) which is independent of m_t , the W boson can be considered again as heavy. However, since there is only one GIM subtraction, both W boson propagators cannot be removed directly. Instead the heavy W limit can be realized by following the steps leading to Eq. (7), i. e.

$$\begin{aligned} \int \frac{dp^2}{(p^2 + m_c^2)(p^2 + m_W^2)^2} &= \frac{1}{m_W^2 - m_c^2} \int \frac{dp^2}{p^2 + m_W^2} \left[\frac{1}{p^2 + m_c^2} - \frac{1}{p^2 + m_W^2} \right] \\ &\approx \frac{1}{m_W^2} \left[\frac{1}{m_W^2} \int_{m_c^2}^{m_W^2} \frac{dp^2}{p^2} - \frac{1}{m_W^2} \right] \end{aligned}$$

where the first integral receives contributions from $m_c^2 \leq p^2 \leq m_W^2$ and the second one from $p^2 \approx m_W^2$ [2, 4, 5]. Thus we obtain the QCD corrected form

$$H_{WW}^a(c, t) = \frac{1}{m_W^2} \int_{m_c^2}^{m_W^2} \frac{dp^2}{p^2} m_c^2(p^2) Q_1(p^2) Q_2(p^2) - m_c^2(m_W^2) Q_2(m_W^2) \quad (20)$$

This was already derived in ref. [5], where the number of flavours, however, was treated in an approximate way as discussed in the last section. Following section 2 (see Eqs. 11 - 17) a rigorous result can now be obtained from Eq. (20). The integral is the same as Eq. (11) with m_t replaced by m_W and some obvious readjustments of the number of flavours. After these minor modifications the result agrees completely with the corresponding formula of ref. [6] which, however, neglects the second term in Eq. (20) (see below for further discussions on this point).

For Eq. (19), however, the heavy W limit is no longer justified. In fact as noted in ref. [5], the corrections in Eq. (3) do not arise any more, because the diagrams with the gluon connecting an external with an internal quark line decouple in this case. Then Eq. (19) with QCD corrections becomes

$$H_{WW}^b(c, t) \approx - \int \frac{dp^2 m_c^2(p^2) m_W^2}{(p^2 + m_t^2(p^2))(p^2 + m_W^2)^2} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \quad (21)$$

Here only the leading order term in m_c^2 is retained. For accurate result the integration in Eq. (21) is done numerically considering the region $p^2 > \mu^2$ only. In course of the integration the number of active flavours in the QCD corrections should be adjusted according to the value of p^2 as discussed in section 2.

An analytical formula can be easily obtained by the replacement $m_t^2(p^2) \approx m_t^2$ in Eq. (21). This corresponds to the approximation $(m_t^2(p^2) - m_t^2)/(p^2 + m_t^2) \ll 1$. Since for the above integral the dominant contribution comes from $m_W^2 \leq p^2 \leq m_t^2$, this approximation is reasonable. Now splitting the integrand such that different terms receive contributions from different regions of p^2 [2, 4, 5] we obtain

$$\begin{aligned} H_{WW}^b(c, t) &\approx \int \frac{dp^2 m_c^2 m_W^2}{m_t^2 - m_W^2} \left[\frac{1}{(p^2 + m_t^2)(p^2 + m_W^2)} - \frac{1}{(p^2 + m_W^2)^2} \right] \\ &\quad \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)} \right)^{8/b(n_3)} \\ &\approx \frac{x_c}{x_t - 1} \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{10/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{10/b(5)} \\ &\quad \left[\frac{1}{x_t - 1} J(m_t^2, m_W^2, \frac{10}{b_5}) - 1 \right] \end{aligned} \quad (22)$$

As already discussed (see e. g. Eq. (9)) the second term in the square bracket

of (22) can be obtained by evaluating the QCD corrections at $p^2 \approx m_W^2$. The first term, however, should be evaluated as in Eqs.7, 13 and 14. This procedure leads to the approximate result of eq. (22).

In ref. [6] QCD corrections in the interval $m_W^2 \rightarrow m_t^2$ were neglected. In this approximation ($\alpha(p^2) \approx \alpha(m_W^2)$ for $p^2 > m_W^2$), J in Eq. (22) reduces to $\ln x_t$ and the result of ref. [6] is regained. The evolution above the scale m_W is given by J . However, J multiplies a term suppressed by $x_t - 1$ and does not affect the numerical result appreciably for $m_t = 174 \text{ GeV}$. It should be noted that the new correction is not an overall multiplicative factor. This feature can be seen in all the subsequent formulae.

In ref. [7] it was suggested that QCD corrections to the ct graph can be computed by substituting $m_W \approx m_t$ in the corresponding formula of ref. [3]. In view of Eq.(20) and the discussions following it, it is clear that this is justified for the part of H_{eff} given in Eq. (18), apart from the neglect of the second term in Eq.(20). We have checked that this corresponds to an error of 6 %. This prescription is obviously not justified for the piece in Eq.(19). However, the contribution of Eq. (19) is numerically much smaller than that of Eq. (18). A posteriori the approximation of [7] appears to be reasonable.

Following the same procedure as above one obtains (keeping only the leading order term in m_c^2) for HH and WH exchanges in the ct diagram:

$$H_{HH}(c, t) \approx \frac{m_c^2 m_t^2}{4m_W^2} \int dp^2 \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \frac{p^2}{(p^2 + m_t^2(p^2))(p^2 + m_W^2)^2} \quad (23)$$

and

$$H_{WH}(c, t) \approx 2m_c^2 m_t^2 \int dp^2 \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \frac{1}{(p^2 + m_t^2(p^2))(p^2 + m_W^2)^2} \quad (24)$$

The approximation for performing the integrals analytically has already been described . Introducing this we arrive at the results

$$H_{HH}(c, t) = \frac{m_c^2 m_t^2}{4m_W^2} \int dp^2 \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)}$$

$$\begin{aligned}
& \left[\frac{-m_W^2}{(m_t^2 - m_W^2)(p^2 + m_W^2)^2} + \frac{m_t^2}{(m_t^2 - m_W^2)(p^2 + m_t^2)(p^2 + m_W^2)} \right] \\
& \approx \frac{x_c x_t}{4} \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{10/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{10/b(5)} \\
& \left[-\frac{1}{x_t - 1} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{8/b(5)} \right. \\
& \quad \left. + \frac{x_t}{(x_t - 1)^2} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{8/b(5)} J(m_t^2, m_W^2, \frac{18}{b(5)}) \right], \quad (25)
\end{aligned}$$

and

$$\begin{aligned}
H_{WH}(c, t) &= 2m_c^2 m_t^2 \int dp^2 \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_c^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{8/b(n_3)} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \\
& \left[\frac{1}{(m_t^2 - m_W^2)(p^2 + m_W^2)^2} \right. \\
& \quad \left. - \frac{1}{(m_t^2 - m_W^2)(p^2 + m_t^2)(p^2 + m_W^2)} \right] \\
& \approx \frac{2x_c x_t}{(x_t - 1)} \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{10/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{10/b(5)} \\
& \left[\left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{8/b(5)} - \frac{1}{x_t - 1} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{8/b(5)} J(m_t^2, m_W^2, \frac{18}{b(5)}) \right] \quad (26)
\end{aligned}$$

As has already been stated, they give good estimates, but for very accurate computation it is desirable to integrate the exact equations numerically. The QCD corrections from $m_t \rightarrow m_W$ are contained in the square brackets. The three overall factors give the QCD evolution from $\mu \rightarrow m_W$, and agrees with ref. [6, 8].

Finally, we consider the $t\bar{t}$ graphs. For reasons discussed earlier, the correction in Eq. (3) are no longer relevant. Including the other two types of corrections we obtain for the WW, HH and WH exchanges

$$\begin{aligned}
H_{WW}(t, t) &= m_W^2 \int dp^2 m_t^4(p^2) \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \\
& \frac{1}{(p^2 + m_t^2(p^2))^2 (p^2 + m_W^2)^2} \quad (27)
\end{aligned}$$

$$H_{HH}(t, t) = \frac{m_t^4}{4m_W^2} \int dp^2 \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{16/b(n_3)} \frac{p^4}{(p^2 + m_t^2(p^2))^2 (p^2 + m_W)^2} \quad (28)$$

$$H_{WH}(t, t) = 2m_t^4 \int dp^2 \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{16/b(n_3)} \frac{p^2}{(p^2 + m_t^2(p^2))^2 (p^2 + m_W)^2} \quad (29)$$

The two types of QCD corrections arise from the running top quark mass and the diagrams in fig. 2. They produce the terms containing $b(n_3)$ and $b(n_2)$, respectively. Again the numbers of flavors should be assigned consistently at each momentum interval. Thus these formulae with the appropriate choice of the number of generations n_2, n_3 and μ can be used for computing the contributions of the box diagrams for K- and B-mesons.

Introducing again the approximation described after Eq. (21) we arrive at the formulae :

$$\begin{aligned} H_{WW}(t, t) &= m_t^4 m_W^2 \int \frac{dp^2}{(m_t^2 - m_W^2)^2} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \\ &\quad \left[\frac{1}{(p^2 + m_t^2)^2} + \frac{1}{(p^2 + m_W)^2} - 2 \frac{1}{(p^2 + m_W^2)(p^2 + m_t^2)} \right], \\ &\approx \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{2/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{2/b(5)} \\ &\quad \left[\frac{x_t}{(x_t - 1)^2} \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_W^2)} \right)^{2/b(5)} + \frac{x_t^2}{(x_t - 1)^2} \right. \\ &\quad \left. - 2 \frac{x_t^2}{(x_t - 1)^3} J(m_t^2, m_W^2, \frac{2}{b_5}) \right], \end{aligned} \quad (30)$$

Special care should be taken in deriving eq. (30) from eq. (27). The $m_t^4(p^2)$ in Eq. (27) originates from the top propagator due to GIM subtraction.

Consequently, when we neglect the running of the top quark mass in the propagator, we must consistently neglect the running of the $m_t^4(p^2)$ -terms. Keeping this in mind we also obtain

$$\begin{aligned}
H_{HH}(t, t) &= \frac{m_t^4}{4m_W^2} \int \frac{dp^2}{(m_t^2 - m_W^2)^2} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{16/b(n_3)} \\
&\quad \left[\frac{m_t^4}{(p^2 + m_t^2)^2} + \frac{m_W^4}{(p^2 + m_W^2)^2} - 2 \frac{m_W^2 m_t^2}{(p^2 + m_W^2)(p^2 + m_t^2)} \right], \\
&\approx \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{2/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{2/b(5)} \\
&\quad \left[\frac{x_t^3}{4(x_t - 1)^2} \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_W^2)} \right)^{2/b(5)} + \frac{x_t^2}{4(x_t - 1)^2} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} - \right. \\
&\quad \left. \frac{x_t^3}{2(x_t - 1)^3} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} J(m_t^2, m_W^2, \frac{18}{b_5}) \right], \tag{31}
\end{aligned}$$

$$\begin{aligned}
H_{WH}(t, t) &= 2m_t^4 \int \frac{dp^2}{(m_t^2 - m_W^2)^2} \left(\frac{\alpha_{n_2}(p^2)}{\alpha_{n_2}(\mu^2)} \right)^{2/b(n_2)} \left(\frac{\alpha_{n_3}(p^2)}{\alpha_{n_3}(m_t^2)} \right)^{16/b(n_3)} \\
&\quad \left[-\frac{m_t^2}{(p^2 + m_t^2)^2} - \frac{m_W^2}{(p^2 + m_W^2)^2} \right. \\
&\quad \left. + \frac{m_t^2 + m_W^2}{(p^2 + m_W^2)(p^2 + m_t^2)} \right] \\
&\approx \left(\frac{\alpha_3(m_c^2)}{\alpha_3(\mu^2)} \right)^{2/b(3)} \left(\frac{\alpha_4(m_b^2)}{\alpha_4(m_c^2)} \right)^{2/b(4)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_b^2)} \right)^{2/b(5)} \frac{2x_t^2}{(x_t - 1)^2} \\
&\quad \left[\frac{(1 + x_t)}{(x_t - 1)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} J(m_t^2, m_W^2, \frac{18}{b(5)}) \right. \\
&\quad \left. - \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{-2/b(5)} - \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} \right]. \tag{32}
\end{aligned}$$

We note that the three overall factors in Eqs. (30) - (32) give the QCD evolution from $m_W \rightarrow \mu$ as was already noted in ref. [6, 8]. The results of ref. [6] are obtained by setting $\alpha(m_t^2) = \alpha(m_W^2)$ and $J = \ln x_t$ in Eq. (30) - (32).

For $B - \bar{B}$ mixing only tt diagrams are relevant. QCD corrections can be obtained by carrying out the p^2 integrations in Eqs. (27)–(29) numerically. However, in this case $\mu = 0(m_B)$. Hence only $p^2 > m_b^2$ should be considered. The analytical formulae in this case are

$$H_{WW}(t, t) \approx \left(\frac{\alpha_5(m_W^2)}{\alpha_5(\mu^2)} \right)^{2/b(5)} \frac{x_t}{(x_t - 1)^2} \left[\left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_W^2)} \right)^{2/b(5)} + x_t - 2 \frac{x_t}{(x_t - 1)} J(m_t^2, m_W^2, \frac{2}{b_5}) \right], \quad (33)$$

$$H_{HH}(t, t) \approx \left(\frac{\alpha_5(m_W^2)}{\alpha_5(\mu^2)} \right)^{2/b(5)} \frac{x_t^2}{4(x_t - 1)^2} \left[x_t \left(\frac{\alpha_5(m_t^2)}{\alpha_5(m_W^2)} \right)^{2/b(5)} + \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} - \frac{2x_t}{(x_t - 1)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} J(m_t^2, m_W^2, \frac{18}{b_5}) \right], \quad (34)$$

$$H_{WH}(t, t) \approx \left(\frac{\alpha_5(m_W^2)}{\alpha_5(\mu^2)} \right)^{2/b(5)} \frac{2x_t^2}{(x_t - 1)^2} \left[\frac{(1 + x_t)}{(x_t - 1)} \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} J(m_t^2, m_W^2, \frac{18}{b_5}) - \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{-2/b(5)} - \left(\frac{\alpha_5(m_W^2)}{\alpha_5(m_t^2)} \right)^{16/b(5)} \right]. \quad (35)$$

We are now prepared to discuss the numerical results.

4 Numerical Results and Conclusions

The main results of this article are QCD corrected effective Hamiltonians in the LLA for $K^0 - \bar{K}^0$ and $B^0 - \bar{B}^0$ systems given by the formulae (23)–(35) in the case $m_t > m_W$. We present for the first time analytical formulae which show explicitly the effect of QCD evolutions in the interval $m_W \rightarrow m_t$, neglected by other authors. Further we have shown the agreement between different calculations which apparently use different methods.

All parameters in the formulae are defined except for the infrared scale μ , which is a subtraction point. For physical states the low energy scale

must be set by the masses of the quarks and the dynamics of confinement. A reasonable expectation is $\mu = 0(m_K)$ for K-mesons and $\mu = 0(m_B)$ for B-mesons. This, of course, means that the renormalization for B-mesons is stopped at M_B , while for K-mesons is continued down to and below the m_c threshold. For this reason formulae for the two cases are different: for B-mesons we do not cross the m_c threshold.

The μ -dependence brings in a factor $(\alpha_{n_2}(\mu^2))^{-2/b(n_2)}$ which should be cancelled, in principle, by a corresponding factor in the hadronic matrix element. This motivated some authors [8, 9] to factor out such a term and define the QCD factors without it. This does not solve the problem but pushes it to another part of the theory; up to now there is no explicit calculation where the μ -dependence of the matrix element cancels the μ -dependence from QCD. For the K-meson we will take $\alpha_s(m_K^2) \approx 0.29$ and then take the 6/25 root of it, which brings the μ -term closer to 1. For B-mesons we will take $\mu = 0(m_B)$.

We use the formulae of section 3 to calculate the integrals for various values of a heavy top quark. We present our results in several tables. Table 1 shows the corrections for the tt-graphs of $K^0 - \bar{K}^0$ mixing for $m_t = 174\text{GeV}$, $m_c = 1.3\text{GeV}$, $m_b = 4.49\text{GeV}$, $m_W = 80\text{GeV}$ and $\mu = 0.50\text{GeV}$. Throughout the paper the Λ parameter of QCD with three active flavours will be taken as input with a value $\Lambda_3 = 0.3$ which is consistent with the value of Λ_5 in [12]. This parameter for other numbers of active flavours will be determined from the input value via the matching conditions [3]. We show three terms of the Hamiltonian with WW, WH and HH exchanges, respectively. The first column is the box diagram with only weak terms. The second column gives the values when QCD corrections are included and the integrals in Eqs (27) - (29) are evaluated numerically. As we discussed in the previous section, the evolution of $\alpha_s(p^2)$ for $p^2 > m_W^2$ is now included. We notice that the change from the first to the second column is large, giving an

$$\eta_2 = 0.56.$$

This means that, as already noted by previous works, the strong contributions are sizeable and must be included.

In the previous sections, we also demonstrated that we can calculate the integrals analytically by introducing an approximation. We calculated these terms and found out that they are in reasonable agreement with the numbers obtained by numerical integration. The analytical formulae are

particularly useful for studying the evolution above m_W because the effects of this evolution is explicitly separated. The separation shows that the evolution above m_W is not quite negligible when corrections to individual terms in Eqs. (30) - (32) are considered. For the purpose of illustration we show below the numerical values of the QCD corrections in Eq. (31).

$$H_{HH}(t, t) \approx 0.58 \frac{x_t^2}{4(x_t - 1)^2} [0.97x_t + 1.29 - 1.12 \frac{2x_t}{x_t - 1} \ln x_t] , \quad (36)$$

In Eq. (36) the pure electroweak term is kept unmodified, i.e. setting all numerical factors equal to one we recover the electroweak term. The factor 0.58 is due to the QCD correction in the interval $\mu^2 \rightarrow m_W^2$ [6, 7, 8]. The effects of the evolution in the interval $m_W \rightarrow m_t$ are shown explicitly as a modification of the coefficients in the square bracket. The bulk of these effects comes from the running of the top mass which in this case has significantly larger anomalous dimension compared to the corrections in Eq. (4). The top quark mass correction is absent in the first term in Eq. (36) since this term is dominated by $p^2 \approx m_t^2$ (see the first step of Eq. (31)). The second term which is dominated by $p^2 \approx m_W^2$ (see the first step of Eq. (31)) receives the maximum correction (1.29), as expected. But the overall numerical impact of this term is rather modest since it multiplies a small electroweak term. The complete results are shown in table 2. The first column gives the results from the full analytical formulae(Eqs. (30) - (32)) while the second column is obtained by ignoring the evolutions above m_W . We also performed the numerical integration in the latter approximation (i.e, by setting $\alpha(p^2) = \alpha(m_W^2)$ for $m_W^2 < p^2$). The results are given in the third column of table 1 and are in fair agreement with the analytical results. The corrections for the H_{WW} and H_{HH} terms differ from those on the second column, but the deviations are in opposite directions so that they partly cancel in the sum.

In Table 3 we show the results for the ct-diagrams for $m_t = 174 GeV$. We note that the largest contribution comes from the H_{WW}^a term (Eq. (20)) whose integral is dominated by the region $m_c^2 \leq p^2 \leq m_W^2$. All other terms are smaller. We obtain the correction η_3 in agreement with previous results.

Table 4 shows the corrections for the tt-graphs of the K-mesons for $m_t = 150$ and $200 GeV/c^2$. We show again three terms of the Hamiltonian with

WW, WH and HH exchanges. We emphasize that the values for η_2 are the same demonstrating, a posteriori, that a precise definition of the running m_t mass is not crucial as long as it is within the range permitted by the experiments [10, 11].

We did not calculate the ct-graphs for other values of m_t because the dominant term by far is H_{WW}^a which is independent of m_t . In summary we see that the QCD corrections to box diagrams of the K-system are well understood and give precise values. The values for the $K^0 - \bar{K}^0$ system are

$$\eta_2 = 0.56 \quad \text{and} \quad \eta_3 = 0.36 .$$

We computed the effective Hamiltonians for the B-system also for three masses of the top quark $m_t = 150, 174$ and 200 GeV and for the parameters $m_c = 1.3 \text{ GeV}$, $m_b = 4.49 \text{ GeV}$, $m_W = 80 \text{ GeV}$ and $\mu = 4.49 \text{ GeV}$. In Table 5 we show the three terms of the Hamiltonian for WW, WH and HH exchanges, separately. We note that η_2 is now larger than the K system.

$$\eta_2 = 0.81$$

and stable on various mass of m_t . A large part of the change comes from the change of the scale μ . In the last column of Table 5 we give the results for a hypothetical T quark belonging to the fourth generation. In this case η_2 changes perceptibly. Ignoring the evolution in the interval $m_W \rightarrow m_T$ one obtains $\eta_2 = 0.84$. We have also verified that the analytical formulae Eqs. (33) - (35) give numbers in agreement with Table 5.

We can compare these results with other calculations.

1. For the K-system. The value of η_2 is practically the same as in our earlier paper [5]. For η_2 there is a next-to-leading order calculation [8] which gives again the same value. We have shown in this article that η_2 is stable under variations of the top quark mass between 150 and 200 GeV. For η_3 the new value is in agreement with our earlier result [5].
2. For the B-system the dominant contribution comes from the tt-diagrams. The η_2 is larger than in the K^0 -mesons. This comes from the change in the μ -scale and the higher threshold. In comparing with ref. [8] one must be careful to correct for the $(\alpha_{\eta_2}(\mu^2))^{2/b(\eta_2)}$ which is included in our formulae. In ref. [8] this factor is extracted and consequently the value for η_{2B} is practically equal to η_{2K} .

In conclusion, the QCD factors which enter calculations with box diagrams have stabilized over the past few years. Improvements through the inclusion of thresholds and variations on the mass of the top quarks discussed in these papers change the values very little or not at all. The QCD factors are much more stable and better understood than the other parameters, in particular the Cabibbo-Kobayashi-Maskawa matrix elements and reduced matrix elements, which enter these calculations.

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	without	with QCD numerically	with QCD negl. above m_W
H_{WW}	0.61	0.37	0.36
H_{WH}	1.25	0.72	0.73
H_{HH}	0.72	0.36	0.42
sum	2.58	1.45	1.51
$\eta_2 =$		0.56	0.59

Table 1: QCD corrections for the tt-diagrams for $m_t = 174\text{GeV}/c^2$

	with QCD(A)	with QCD(B)
H_{WW}	0.36	0.35
H_{WH}	0.77	0.72
H_{HH}	0.34	0.42
sum	1.47	1.49
$\eta_2 =$	0.57	0.58

Table 2: Corrections for the tt-diagrams for $m_t = 174\text{GeV}/c^2$. A) Results from full analytical formulae. B) Results after neglecting the evolution above m_W .

	without	with QCD numerically	with QCD negl. above m_W
H_{WW}^a	1.91	0.78	0.78
H_{WW}^b	-0.041	-0.006	-0.007
H_{WH}	0.39	0.072	0.074
H_{HH}	0.081	0.012	0.014
sum	2.34	0.86	0.86
$\eta_3 =$		0.37	0.37

Table 3: Corrections for the ct-diagrams in units 10^{-3} for $m_t = 174\text{GeV}/c^2$

	For $m_t = 150\text{GeV}/c^2$		For $m_t = 200\text{GeV}/c^2$	
	without	with QCD numerically	without	with QCD numerically
H_{WW}	0.56	0.34	0.65	0.40
H_{WH}	1.00	0.58	1.50	0.87
H_{WW}	0.49	0.24	1.02	0.51
sum	2.05	1.16	3.17	1.78
$\eta_2 =$	0.57		0.56	

Table 4: The tt-case in units of 1 for values of m_t

	$m_t = 150\text{GeV}/c^2$		$m_t = 174\text{GeV}/c^2$		$m_t = 200\text{GeV}/c^2$		$m_t = 500\text{GeV}/c^2$	
	without	with QCD	without	with QCD	without	with QCD	without	with QCD
H_{WW}	0.56	0.49	0.61	0.53	0.65	0.57	0.87	0.74
H_{WH}	1.00	0.84	1.25	1.04	1.50	1.26	3.91	3.17
H_{HH}	0.49	0.35	0.72	0.52	1.02	0.74	8.55	6.51
sum								
$\eta_2 =$	0.82		0.81		0.81		0.74	

Table 5: The tt-case for B-mesons in units of 1